On the Negative Case of the Singular Yamabe Problem

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Abstract: Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$, and let Γ be a nonempty closed subset of M. The negative case of the Singular Yamabe Problem concerns the existence and behavior of a complete metric $\hat{g}$ on $M \backslash \Gamma$ that has constant negative scalar curvature and is pointwise conformally related to the smooth metric g. Previous results have shown that when Γ is a smooth submanifold of dimension d there exists such a metric if and only if $d > \frac{n-2}{2}$. In this paper, we consider a general class of closed sets and show the existence of a complete conformal metric $\hat{g}$ with constant negative scalar curvature depends on the dimension of the *tangent cone to* Γ at every point. Specifically, provided Γ admits a nice tangent cone at p, we show that when the dimension of the tangent cone to Γ at p is less than $\frac{n-2}{2}$ then there can not exist a *negative Singular Yamabe metric* $\hat{g}$ on $M \backslash \Gamma$.

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1. Introduction:

The resolution of the *Yamabe Problem* by Schoen (cf. [LP] or [Sc1]) showed that every compact Riemannian manifold of dimension $n \geq 3$ is conformally equivalent to one with constant scalar curvature. A natural question to then ask is whether every noncompact Riemannian manifold of dimension $n \geq 3$ is conformally equivalent to a complete manifold with constant scalar curvature. For noncompact manifolds with a simple structure at infinity, this question may be studied by solving the so-called *Singular Yamabe problem*:

> Given a compact Riemannian manifold (M, g) of dimension $n \geq 3$, and a nonempty closed set Γ in M. Find a complete metric $\hat{g}$ on $M \backslash \Gamma$ with constant scalar curvature that is pointwise conformally related to g.

The principal benefit of studying this version of the Yamabe Problem on noncompact manifolds is that the complete metric $\hat{g}$ has a uniform underlying structure at infinity. The general problem of conformally deforming an arbitrary metric $\bar{g}$ to a complete metric $\hat{g}$ with constant scalar curvature requires extra conditions on the metric structure of $\bar{g}$ at infinity (for example see [LTY]).

In this paper, we consider the *negative case*, when the complete metric $\hat{g}$ is to have constant negative scalar curvature. We shall not attempt to give a survey of the results in either the positive case or the flat case. Instead, we direct the reader to the introductions of the recent papers [MP], [MPU], and the references contained therein. We only note that a major difference between the negative case and the positive case (and the flat case) is the size of Γ for which there exists a Singular Yamabe metric. For example, suppose Γ is a smooth submanifold of dimension d in $\mathbf{S}^n$. The results in [Av], [De], [LN], [Mz], [SY1], [SY2] then show there can exist a complete metric with constant positive scalar curvature or zero scalar curvature only if $d \leq \frac{n-2}{2}$, and there can exist a complete metric with constant scalar negative curvature only if $d > \frac{n-2}{2}$.

The first results on the negative case appeared in Loewner and Nirenberg's seminal paper [LN] on partial differential equations invariant under conformal transformations. In that paper, Loewner and Nirenberg showed when Γ is a smooth submanifold of dimension d in $\mathbf{S}^n$ there exists a negative Singular Yamabe metric $\hat{g}$ on $\mathbf{S}^n \backslash \Gamma$ if $d > \frac{n-2}{2}$, and there

does not exists such a metric if the Hausdorff dimension of Γ is less than $\frac{n-2}{2}$. Subsequent work by Aviles and McOwen in [AM2] has since generalized the existence result in [LN] to an arbitrary compact manifold proving that there exists a negative Singular Yamabe metric on $M\backslash\Gamma$ when Γ is a smooth submanifold of dimension d if and only if $d > \frac{n-2}{2}$. This necessary and sufficient condition for existence has been recently extended to smooth submanifolds of dimension d with boundary (cf. [Fn1]).

The original purpose of this work was to examine how singularities in the structure of Γ affect the behavior of a negative Singular Yamabe metric as a prelude to establishing existence. But, in the course of our investigations, it became apparent that singularities not only affect behavior but also existence. Specifically, we found that the dimension of the *tangent cone* affects existence.

Theorem: *Suppose that Γ has a proper tangent cone at p, and that the dimension of this tangent cone is less than $\frac{n-2}{2}$. Then there can not exist a negative Singular Yamabe metric on $M\backslash\Gamma$.*

We will define a proper tangent cone and discuss the dimension of a tangent cone in the next section. However, to understand the significance of our result, the formal definitions of these terms are unnecessary. For instance, consider a singular hypersurface Γ that is locally (near some point) of the form

$$\{(x_1, \cdots, x_n) : x_1^2 + \cdots + x_k^2 = x_n^3\}.$$

Our main result then shows that there can not exist a negative Singular Yamabe metric on $M\backslash\Gamma$ when $k > \frac{n+2}{2}$. (The tangent cone to such a set at the origin is an upper half plane of dimension $n-k$.) This result is somewhat surprising for this choice of Γ, since after all Γ is a hypersurface and all the previous results pointed towards existence depending only on the "global dimension" of Γ.

We obtain our result by examining the behavior of the maximal positive solution to the following semi-linear elliptic problem:

$$(\dagger) \qquad \begin{cases} \Delta_g u = u^q + Su \quad \text{on} \quad M\backslash\Gamma \\ u(x) \to +\infty \quad \text{as} \quad x \to \Gamma, \end{cases}$$

where Δ_g is the Laplace operator on (M, g), q is an arbitrary constant greater than one, and $S \in C^\infty(M)$. This works because the equation in (†) guarantees that the conformal metric $\hat{g} = u^{q-1}\, g$ has constant negative scalar curvature (cf. [LP] or [Kz]), when $q = \frac{n+2}{n-2}$ and S is a specific multiple of the scalar curvature of g. Furthermore, if we can show that a solution u to (†) tends to infinity at a sufficiently fast rate then the metric $\hat{g}$ will be complete. Conversely, if we can show that the maximal solution to (†) does not tend to infinity at a fast rate then the metric $\hat{g}$ can not be complete, and thus there can not exist a negative Singular Yamabe metric on $M \backslash \Gamma$.

2. Proper Tangent Cones

As stated previously, the original purpose of this paper was to study how singularities in the structure of Γ affect the behavior of the maximal solution to (†) and this the existence of a negative Singular Yamabe metric. We were going to start by considering the case where Γ is a stratified set, that is when Γ can be decomposed into a finite number of open embedded smooth submanifolds. A prime example of such a set is an algebraic variety of $\mathbf{R}^n$, or any set that can locally be obtained (in some coordinate system) as an algebraic variety. This case seemed to be the natural first step after smooth submanifolds before a general closed subset.

Through our investigation of this case, it became apparent that the behavior of the maximal solution to (†) depended on the tangent structure of Γ. Therefore, instead of using the standard conditions on a stratified set (cf. [Wh1], [Wh2], [GM] or [Th]), we developed the following notion of a *proper tangent cone* and placed conditions on the set through the admissibility of a proper tangent structure in [Fn2]. As will become clear through the definitions below, we do not require the set to be stratified for our main result. However, for the purpose of understanding the following definitions, it is useful to restrict our attention to stratified sets, partly because we require a proper tangent cone to admit a stratification by smooth submanifolds.

A stratified set is a set Γ in an ambient manifold that can be decomposed into a locally finite collection of disjoint smooth submanifolds. This collection of smooth submanifolds is called a stratification of Γ, and each submanifold is called a stratum (plural strata). Typically, one also enforces the condition that the stratification of Γ, $\{\Sigma_\alpha\}$, satisfies the *axiom of frontier*: $\Sigma_\alpha \cap \operatorname{clos}(\Sigma_\beta) \neq \emptyset$ implies that $\Sigma_\alpha \subset \operatorname{clos}(\Sigma_\beta)$.

Our first guess for defining a tangent cone to a stratified set Γ at a point p is to generalize the definition of the tangent space to an embedded smooth submanifold as the set of vectors $v \in T_pM$ such that there exists a C^1 path $\gamma\colon [0,1] \to M$ with $\gamma(0) = p$, $\gamma'(0) = v$ and $\gamma(t) \in \Gamma$ for all $t \in [0,1]$. From this definition it follows that if v is a tangent vector to Γ at p then for all $t \geq 0$ the vector tv is also a tangent vector to Γ at p. Therefore, the set of tangent vectors to Γ at p is the cone over a set of unit vectors in T_pM, which

we call the *cone of tangent vectors.* Notice that in this definition, if p is a regular point of Γ, when the set Γ is locally a smooth submanifold near p, the cone of tangent vectors coincides with the usual definition of the tangent space to Γ at p.

We note however that for an arbitrary stratified set Γ this notion of the a cone of tangent vectors does not necessarily contain all the infinitesimal information about p. For example, consider the stratified set in $\mathbf{R}^2$ given by

$$\{(0,t) : -1 \leq t \leq 1\} \bigcup \{(s, \sin(s)) : s > 0\}.$$

Every tangent vector to this set at the origin is of the form $(0, y)$, but every open set in Γ containing the origin also contains points of the form $(x, 0)$. To enlarge our notion of a tangent cone, we define the *essential link of* Γ *at* p as an infinitesimal version of an ϵ-link of Γ at p. In singularity theory, an ϵ-link of Γ at p is defined as the intersection of Γ with a sphere of radius ϵ centered at p, and is used to determine the topological type of the singularity of Γ at p (cf. [Mi]).

We define the essential link of Γ at p by examining the set of unit vectors in T_pM which are mapped into Γ by geodesics. Thus, our definition employs the map from $M \backslash \{p\}$ to T_pM defined by

$$\omega_p(x) = \frac{v}{|v|} \quad \text{if} \quad x = \exp_p(v).$$

This map is well-defined in the punctured ball of radius r_0 about p, $B_{r_0}(p) \backslash \{p\}$, where r_0 is a chosen constant less than the injectivity radius of (M, g). Notice the map ω_p just gives the spherical projection for a point x in a geodesic polar coordinate system based at p. Using this map ω_p, we can view an ϵ-link of Γ at p as a subset of $\mathbf{S}^{n-1}$. Therefore, we can take the limit of the ϵ-links of Γ at p as ϵ tends to zero. The resulting limit set could then be viewed as an infinitesimal version of an ϵ-link. But to guarantee that this limits exists, it is more convenient to define this *essential link of* Γ *at* p as the following subset of the unit sphere in T_pM:

$$L_p\Gamma = \bigcap_{0<\epsilon<r_0} \operatorname{clos}\big(\omega_p(\Gamma \cap B_\epsilon^*)\big)$$

where B_ϵ^* is the punctured ball of radius ϵ, $B_\epsilon(p)\backslash\{p\}$.

This definition of an essential link is closely related to the definition of an tangent cone in geometric measure theory (cf. [Fe] or [Mo]). In fact, in geometric measure theory, the tangent cone of an arbitrary set is defined to be the cone over the essential link. This is the definition of a tangent cone we will be using in this paper. Specifically, we define the *tangent cone of* Γ *at* p as

$$T_p\Gamma = \{w \in T_pM\colon w = tv \text{ with } t \geq 0 \text{ and } v \in L_p\Gamma\}.$$

We note that this cone of tangent vectors to Γ is clearly contained in $T_p\Gamma$, and for most nice stratified sets the cone of tangent vectors is equal to $T_p\Gamma$.

In defining a *proper tangent cone*, we would like to guarantee that both $T_p\Gamma$ and $L_p\Gamma$ satisfy certain regularity conditions. This is because it is possible to construct stratified sets where $L_p\Gamma$ (and hence $T_p\Gamma$) is a Cantor set. Simply notice that the process for defining the essential link is similar to an inductive process. Therefore, it is not hard to construct a stratified set (with infinite topological type) that has an essential link that is a Cantor set. To avoid such sets, we demand that the essential link is a stratified set when defining a proper tangent cone.

> **Definition:** A tangent cone to Γ at a point p is called a *proper tangent cone* if the essential link of Γ at p admits a stratification by smooth submanifolds.

By requiring the essential link to be stratified, we guarantee that a proper tangent cone is also stratified.

We define the dimension of a stratified set Γ to be d is d is the dimension of the largest submanifold in a stratification of Γ. Therefore, we can define the dimension of a proper tangent cone and the dimension of the essential link of a proper tangent cone using this definition of the dimension of a stratified set. From this, it follows that $\dim(T_p\Gamma) = \dim(L_p\Gamma) + 1$, whenever $T_p\Gamma$ is a proper tangent cone. We note this concept of dimension

does not necessarily agree with the Hausdorff dimension of $T_p\Gamma$ as a subset of T_pM. For example, the Hausdorff dimension of the set

$$\Gamma = \{(0,t) : -1 \leq t \leq 1\} \bigcup \{(s, \sin(s)) : s > 0\} \subset \mathbf{R}^2$$

is two, while the dimension we are using for Γ as a stratified set says the dimension is one.

3. Outline of Proof

Let $\beta_0 = \frac{2}{q-1}$ and $d_0 = n-2-\beta_0$. Notice when $q = \frac{n+2}{n-2}$, we have $\beta_0 = d_0 = \frac{n-2}{2}$. We will first show that if the maximal positive solution u to (†) has a strong singularity at Γ, that is u satisfies the estimate

$$0 < C_1 \le \rho(x)^{\beta_0}\, u(x) \le C_2 < +\infty \tag{3.1}$$

where $\rho(x) = dist_g(x, \Gamma)$ then the asymptotic behavior of u near a point $p \in \Gamma$ can be described in geodesic polar coordinates (r, ω) at p by

$$v(\omega) = \lim_{r \to 0}\; r^{\beta_0}\, u(r, \omega) \tag{3.2}$$

with v being the maximal positive solution to

$$(*) \qquad \begin{cases} \Delta_\omega v = v^q + \beta_0\, d_0\, v \quad \text{on} \quad \mathbf{S}^{n-1} \backslash L_p\Gamma \\ \qquad v(\omega) \to +\infty \quad \text{as} \quad \omega \to L_p\Gamma, \end{cases}$$

where Δ_ω represents the Laplace operator on $\mathbf{S}^{n-1}$ with respect to the standard metric. The fact that u has a strong singularity at Γ implies that v must also have a strong singularity at $L_p\Gamma$; v must satisfy

$$0 < C_1 \le \sigma(\omega)^{\beta_0}\, v(\omega) \le C_2 < +\infty \quad \text{as} \tag{3.3}$$

where $\sigma(\omega) = dist(\omega, L_p\Gamma)$ on $\mathbf{S}^{n-1}$. Such a solution to $(*)$ can only exist if the dimension of $L_p\Gamma$ is greater than $d_0 - 1$ (see Lemma 2 in the next section). Hence, the dimension of the tangent cone $T_p\Gamma$ must be greater than d_0 whenever the maximal solution to (†) has a strong singularity at Γ.

We note that a solution to (†) with a strong singularity at Γ implies that the conformal metric $\hat{g} = u^{q-1}\, g$ is a complete metric. Therefore, we conclude from the above analysis that any obstruction to the existence of a negative Singular Yamabe metric occurs when the dimension of $T_p\Gamma$ is less than or equal to $\frac{n-2}{2}$. We would like to conclude that if the maximal solution does not have a strong singularity then the metric $\hat{g}$ is not complete.

But, our examination of the asymptotic behavior of solutions to (†) does not allow this conclusion. Our analysis is based solely on how the dimension of $T_p\Gamma$ affects the behavior.

We obtain our main result follows from a delicate asymptotic analysis which shows that when the dimension of $T_p\Gamma$ is less than d_0 there exists a $\delta > 0$ and an open path $\gamma\colon (0,1) \to M\backslash\Gamma$ with $p = \lim_{t\to 0} \gamma(t)$ such that the maximal solution to (†) satisfies

$$\limsup_{x\in\gamma} \rho(x)^{\beta_0-\delta}\, u(x) < +\infty. \tag{3.4}$$

The last step in this analysis is to compare the asymptotic behavior of the maximal solution to (†) to a solution of

$$\begin{cases} \Delta_\omega v = \beta_0 d_0\, v \quad \text{on} \quad \mathbf{S}^{n-1}\backslash Ł_p\Gamma \\ v(\omega) \to +\infty \quad \text{as} \quad \omega \to L_p\Gamma. \end{cases} \tag{$*$}$$

This is the step that requires the dimension of $T_p\Gamma$ to be strictly less than d_0. Once we have established (3.4), a simple calculation shows that the metric $\hat{g} = u^{q-1}\, g$ is not complete. Hence, when the dimension of a proper tangent cone is less than $\frac{n-2}{2}$ there can not exist a negative Singular Yamabe metric.

4. Analytical Preliminaries

To obtain our main result, we need the following analytic results on the behavior of positive solutions to $\Delta_g u = u^q + Su$ proved in [Av], [FM] and [Fn1] (see also [Fn2]). The proofs of these results all rely on standard elliptic theory (cf. [GT]). We will also need to use the method of upper and lower solutions, and a global Green's function (cf. [Au], [LP], and [Mc2]).

The first important result on the behavior of positive solutions to (†) is the following *apriori* upper bound found in [Av] and [Vn]

Lemma 1: *Suppose u is a positive solution of $\Delta_g u = u^q + Su$ in $M \backslash \Gamma$. Then*

$$\rho(x)^{\beta_0} \, u(x) \leq C$$

for some positive constant C independent of u and Γ.

In both [Av] and [Vn], this result is stated for the case where Γ is a smooth submanifold, but the proofs are valid for an arbitrary closed set. One of the first uses of this apriori upper bound is the following nonexistence result proved in [Av] and [Vn], and extended in [Fn1], [Fn2] to the following form:

Lemma 2: *Suppose p is a regular point of Γ and the dimension of Γ at p is less than or equal to d_0. Then there exists no positive solution of $\Delta_g u = u^q + Su$ in $M \backslash \Gamma$ satisfying $u(x) \to \infty$ as $x \to p$.*

This result was originally stated for smooth submanifolds, but by introducing a cut-off function on Γ around p into the Aviles' proof one obtains the above generalization (see [Av], [Fn1], [Fn2]). The regularity on Γ is needed to establish a nice coordinate system about Γ, so one may estimate integrals near Γ.

We obtain more refined results on the asymptotic behavior of the maximal solution to (†) from a *Harnack-type inequality* and *derivative estimates* for $\Delta_g u = u^q + Su$ (cf. [FM] and [Fn2]). These estimates are obtained directly from standard elliptic theory as a consequence of the apriori upper bound. Hence, these results do not depend on the regularity of the Γ.

Lemma 3: *Let u be a positive solution to $\Delta_g u = u^q + Su$ in $M\backslash\Gamma$ and x_0 be a fixed point in $M\backslash\Gamma$. Then there exists a positive constant C_1 independent of u, x_0 and Γ such that*

$$\sup_{y\in B} u(y) \le C_1 \inf_{y\in B} u(y)$$

where $B = \{y \in M\backslash\Gamma : dist_g(x_0, y) \le \frac{1}{8}\rho(x_0)\}$.

To state the derivative estimates, it will be convenient to introduce a weighted norm on functions in $C^{2,\alpha}(\Omega)$ where Ω is a smooth submanifold of dimension n with (smooth) boundary in M. Let $d_\xi = dist_g(\xi, \partial\Omega)$ and $d_{\xi,\eta} = \min(d_\xi, d_\eta)$. Furthermore, let $\mathcal{D}u(\xi)$ and $\mathcal{D}^2u(\xi)$ represent the covariant derivative of u and $\mathcal{D}u$ with respect to an orthonormal frame at ξ. We then define the norm $|\cdot|^*_{2,\alpha}$ of a function $u \in C^{2,\alpha}(\Omega)$ by

$$|u|^*_{2,\alpha;\Omega} = |u|^*_{2,0;\Omega} + [u]^*_{2,\alpha;\Omega} \tag{4.1}$$

where

$$|u|^*_{2,0;\Omega} = \sup_{\xi\in\Omega} |u(\xi)| + \sup_{\xi\in\Omega} d_\xi\, |\mathcal{D}u(\xi)| + \sup_{\xi\in\Omega} d_\xi^2\, |\mathcal{D}^2u(\xi)| \tag{4.2}$$

and

$$[u]^*_{2,\alpha;\Omega} = \sup_{\xi,\eta\in\Omega} d_{\xi,\eta}^{2+\alpha}\, \frac{|\mathcal{D}^2u(\xi) - \mathcal{D}^2u(\eta)|}{|\xi-\eta|_g^\alpha}. \tag{4.3}$$

In (4.2), $|\mathcal{D}u(\xi)|$ is the length of the covariant derivative of u at ξ, and $|\mathcal{D}^2u(\xi)|$ is the length of the covariant derivative of $\mathcal{D}u$ at ξ. In (4.3), the addition of vectors in different tangent spaces is accomplished by using parallel transport.

Lemma 4: *Let u be a positive solution to $\Delta_g u = u^q + Su$ on $M\backslash\Gamma$ and let x_0 be a fixed point in $M\backslash\Gamma$. Then there exists a positive constant C_2 independent of x_0 and Γ such that*

$$|u|^*_{0,2;B} \le C_2 |u|_B$$

where $B = \{y \in M\backslash\Gamma : dist_g(x_0, y) \le \frac{1}{8}\rho(x_0)\}$.

From these derivative estimates, we derived the uniqueness of a positive solution to (†) with a strong singularity at Γ in [Fn1].

Lemma 5: *There exists at most one positive solution to* (†) *with a strong singularity at* Γ.

The proof of this uniqueness also requires the Asymptotic Maximum Principle of Cheng and Yau (cf. [CY] and [Au]).

Probably, the most unusual aspect of our nonexistence proof is the use of existence methods. Specifically, we make use of the method of upper and lower solutions and the existence of a global Green's function. We recall that in the method of upper and lower solutions one looks for an upper solution φ, a function that satisfies

$$L[\varphi] + f(x, \varphi) \leq 0,$$

and a lower solution ψ, a function that satisfies

$$L[\psi] + f(x, \psi) \geq 0$$

, such that $\varphi \geq \psi$. A monotone iteration scheme then guarantees the existence of a solution u of $L[u] + f(x, u) = 0$ which satisfies $\psi \leq u \leq \varphi$. However, in our proof, it is necessary to work with Holder continuous upper and lower solutions, instead of the usual C^2 functions. Therefore, we use Calabi's extension of the Hopf maximum principle in the method of upper and lower solutions (cf. [Ca]). This extension is based on the following definition for a continuous function to weakly satisfy a differential inequality.

> **Definition:** Let L be a uniformly elliptic operator of divergence type. We say a continuous function u in an open domain Ω satisfies
>
> $$L[u] \geq v \quad \text{weakly in any subset } K \text{ of } \Omega$$
>
> if for each point $x_0 \in K$ and any given positive constant ϵ there exists an open neighborhood $V_{x_0,\epsilon} \subset \Omega$ containing x_0 and a C^2 function $u_{x_0,\epsilon}$ such that the difference function $u - u_{x_0,\epsilon}$ achieves it minimum value at x_0 and $u_{x_0,\epsilon}$ satisfies the inequality
>
> $$L[u_{x_0,\epsilon}] \geq v - \epsilon \quad \text{in} \quad V_{x_0,\epsilon}$$

in the usual sense. Similarly, we say that $L[u] \leq v$ weakly if $L[-u] \geq -v$ weakly, as defined above.

In fact from the methods in [AM1], we can reduce the method of upper and lower solutions for (†) to a search for nonnegative weak lower solution w that satisfies $w(x) \to +\infty$ as $x \to \Gamma$. (see also [Fn2])

As noted in the outline of our proof, we compare the asymptotic behavior of the maximal solution to (†) to a solution of the linear equation$\Delta_\omega v = \beta_0 d_0 v$ on $\mathbf{S}^{n-1} \backslash L_p\Gamma$. We construct solutions to this linear equation by using the global Green's function to $-\Delta_\omega + \beta_0 d_0$ on $\mathbf{S}^{n-1}$ to define a Poisson transform (cf. [Fn1]). Specifically, there exists a global Green's function $\mathcal{G}(x,y)$ when $d_0 > 0$ and this Green's function satisfies the estimates

$$C_1 \, |x-y|_\omega^{3-n} \leq \mathcal{G}(x,y) \leq C_2 \, |x-y|_\omega^{3-n}$$

when $n \geq 4$, and

$$C_1 \left(1 + \big| \log |x-y|_\omega \big|\right) \leq \mathcal{G}(x,y) \leq C_2 \left(1 + \big| \log |x-y|_\omega \big|\right)$$

when $n = 3$, where C_1, C_2 are positive constants (depending on n and q), and $|x-y|_\omega$ is the distance between x and y on $\mathbf{S}^{n-1}$. Recall that for every $yin\mathbf{S}^{n-1}$, the function $u(x) = \mathcal{G}(x,y)$ is the positive distributional solution of $-\Delta_\omega v + \beta_0 d_0 v = \delta_y$ with δ_y is the Dirac delta distribution at y. With this Green's function, we define a Poisson transform on $\mathbf{S}^{n-1} \backslash L_p\Gamma$ by

$$T[\varphi](x) = \int\limits_{L_p\Gamma} \mathcal{G}(x,y) \, \varphi(y) \, d\nu(y)$$

where $\nu(y)$ is a measure on $L_p\Gamma$ compatible with the induced Riemannian measure on each stratum. Notice for Lebesgue integrable functions (with respect to the measure ν) the Poisson transform constructs classical solutions of $\Delta_\omega v = \beta_0 d_0 v$ on $\mathbf{S}^{n-1} \backslash L_p\Gamma$.

5. Asymptotic Behavior of a Strong Singularity

In this section, we complete the first half of the proof of our main result by examining the asymptotic behavior of a solution to †) with a strong singularity. In particular, we prove

Theorem A: *Suppose that Γ has a proper tangent cone at p, and let u be the unique positive solution to* ($\dagger$) *with a strong singularity at Γ. Then the dimension of $T_p\Gamma$ must be greater than d_0. Moreover, the function v defined by* (3.2) *is the unique positive solution to* ($*$) *with a strong singularity at $L_p\Gamma$.*

This theorem describes the unique solution to ($\dagger$) with a strong singularity at Γ in terms of another unique solution with a strong singularity. Therefore, we may use this theorem inductively, to prove that very strong geometric conditions are needed to prove the existence of a solution to ($\dagger$) with a strong singularity at Γ, provided the essential link $L_p\Gamma$ satisfies sufficient regularity conditions. Consequently, any existence proof for a negative Singular Yamabe metric becomes much more complicated; the previous existence proofs are all based on proving the existence of a positive solution to †) with a strong singularity at Γ, since a solution with a strong singularity guarantees that the conformal metric $\hat{g} = u^{q-1}\, g$ is a complete metric.

We will first establish some notation before proving Theorem A. Let (r, ω) be a geodesic coordinate system centered at p. In terms of this coordinate system, the operator Δ_g takes on the form:

$$\Delta_g = \partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_\omega + \mathcal{L} \tag{5.1}$$

where $\mathcal{L}$ is a second order perturbation operator of form

$$\mathcal{L} = \mathcal{O}(r)\,\partial_r + \mathcal{O}(r^{-1})\,\partial_\omega^2 + \mathcal{O}(r^{-1})\,\partial_\omega. \tag{5.2}$$

In (5.2), we use $\mathcal{O}(r^k)$ to mean the term is dominated by $C\,r^k$ as $r \to 0$, i.e. big-oh notation. Since we will constantly be using expressions similar to (3.1) and (3.3), we introduce the following notation

$$f(r) \approx g(r) \quad \text{as} \quad r \to 0$$

to mean that there exist fixed positive constants C_1, C_2 such that

$$C_1\, f(r) \leq g(r) \leq C_2\, f(r) \quad \text{as} \quad r \to 0.$$

Thus, a solution to (†) with a strong singularity can be described as $u(x) \approx \rho(x)^{-\beta_0}$.

The proof of Theorem A is divided into three propositions concerning the behavior of the following two functions

$$\begin{cases} \overline{v}(\omega) = \limsup\limits_{r\to 0}\ r^{\beta_0}\, u(r,\omega) \\ \underline{v}(\omega) = \liminf\limits_{r\to 0}\ r^{\beta_0}\, u(r,\omega) \end{cases}$$

which will show that $\overline{v} \equiv \underline{v}$, and thus that the function defined by (3.2) exists. These propositions together prove that v is the unique positive solution of $(*)$ with a strong singularity at $L_p\Gamma$, from which Theorem A follows by appealing to Lemma 2 in the previous section.

Proposition 1: $\overline{v},\ \underline{v} \in C^{0,\alpha}(\mathbf{S}^{n-1}\backslash L_p\Gamma)$ *for some* $\alpha \in (0,1)$.

Proof: Choose $\Omega \subset\subset \mathbf{S}^{n-1}\backslash L_p\Gamma$, and define $\Omega(r) = \{x \in M : x = \exp_p(rv) \text{ where } v \in \Omega\}$. Then there exists $\delta > 0$ and $\epsilon > 0$ sufficiently small such that $dist_g(\Omega(r), \Gamma) \geq \epsilon r$ for all $r \in (0,\delta)$, by definition of $L_p\Gamma$. From the apriori upper bound in Lemma 1 there exists a positive constant C depending on ϵ and δ such that

$$v(r,\omega) = r^{\beta_0}\, u(r,\omega) \leq C \quad \text{when} \quad \omega \in \Omega$$

for all $r \in (0,\delta)$. The derivative estimates in Lemma 4 then imply $v_r(\omega) = v(r,\omega) \in C^{2,\alpha}(\overline{\Omega})$ for all $r \in (0,\delta)$ and there exists a positive constant C such that $||v_r||_{2,\alpha;\Omega} \leq C$, (i.e. $||v_r||_{2,\alpha;\Omega}$ is bounded independent of r), where $||\cdot||_{2,\alpha;\Omega}$ is the standard norm on the function space $C^{2,\alpha}(\overline{\Omega})$. Since the derivatives in Ω are bounded independent of r, it follows that the Hölder semi-norm $[v]_{0,\alpha;\Omega}$ is bounded independent of r. Therefore, it follows by standard arguments that $\overline{v}, \underline{v} \in C^{0,\alpha}(\overline{\Omega})$, and thus $\overline{v}, \underline{v} \in C^{0,\alpha}(\mathbf{S}^{n-1}\backslash L_p\Gamma)$. □

Proposition 2: $\overline{v}$ *satisfies* $\Delta_\omega v \geq v^q + \beta_0 d_0 v$ *weakly on* $\mathbf{S}^{n-1}\backslash L_p\Gamma$, *and* $\underline{v}$ *satisfies* $\Delta_\omega v \leq v^q + \beta_0 d_0 v$ *weakly on* $\mathbf{S}^{n-1}\backslash L_p\Gamma$.

Proof: This proposition is proved by appealing to a one-dimensional version of the asymptotic maximum principle, and the definitions of $\overline{v}$ and $\underline{v}$. First, we note that under the change of variables $r = e^{-t}$ and $v(t,\omega) = e^{\beta_0 t}\,u(t,\omega)$ the equation $\Delta_g u = u^q + Su$ may written as

$$\partial_t^2 v + (\beta_0 - d_0)\partial_t v + \Delta_\omega v = v^q + \beta_0 d_0 v + \mathcal{L}v$$

where $\mathcal{L}v = \mathcal{O}(e^{-t})D^2 v + \mathcal{O}(e^{-t})Dv + \mathcal{O}(e^{-2t})v$. With this change of variables, we find

$$\begin{cases} \overline{v}(\omega) = \limsup\limits_{t\to\infty} v(t,\omega) \\ \underline{v}(\omega) = \liminf\limits_{t\to\infty} v(t,\omega). \end{cases}$$

Next, we choose $\Omega \subset\subset \mathbf{S}^{n-1}\backslash L_p\Gamma$ as in Proposition 1, so there exists T sufficiently large such that $v_t(\omega) = v(t,\omega) \in C^{2,\omega}(\overline{\Omega})$ for all $t \in [T,\infty)$ and there exists a positive constant C such that $||v_t||_{2,\alpha} \leq C$ uniformly for all $t \in [T,\infty)$. From a one-dimension version of the asymptotic maximum principle and the definition of $\overline{v}$, we can choose a monotonically increasing sequence $\{T_n\}$ with $T_n \to \infty$ such that

$$\begin{cases} \overline{v}(\omega_0) = \lim\limits_{n\to\infty} v(T_n,\omega_0) \\ \lim\limits_{n\to\infty} \partial_t v(T_n,\omega_0) = 0 \\ \lim\limits_{n\to\infty} \partial_t^2 v(T_n,\omega_0) \leq 0, \end{cases}$$

for any point $\omega_0 \in \Omega$. The Arzela-Ascoli theorem (and the compactness of the embedding $C^{2,\alpha} \to C^2$) implies there exists a subsequence $\{v_m(\cdot) = v(T_{n_m},\cdot)\}$ converging to $v \in C^2(\omega)$. We have $v \leq \overline{v}$ and $v(\omega_0) = \overline{v}(\omega)$. Therefore, for any $\epsilon > 0$, we can find a neighborhood U of ω_0 such that

$$\Delta_\omega v \geq \overline{v}^q + \beta_0 d_0\,\overline{v} - \epsilon \quad \text{in} \quad U.$$

The function $\overline{v} \in C^{0,\alpha}(\mathbf{S}^{n-1}\backslash L_p\Gamma)$ then is a weak lower solution of $\Delta_\omega v = v^q + \beta_0 d_0 v$ as desired.

The proof that $\underline{v}$ is a weak lower solution to $\Delta_\omega v = v^q + \beta_0 d_0 v$ is based on repeating the above argument, using that

$$\begin{cases} \underline{v}(\omega_0) = \lim_{n\to\infty} v(T_n, \omega_0) \\ \lim_{n\to\infty} \partial_t v(T_n, \omega_0) = 0 \\ \lim_{n\to\infty} \partial_t^2 v(T_n, \omega_0) \geq 0 \end{cases}$$

for some monotonically increasing sequence $\{T_n\}$ with $T_n \to \infty$. $\square$

Proposition 3: $\overline{v}(\omega) \approx \underline{v}(\omega) \approx \sigma(\omega)^{-\beta_0}$ as $\sigma(\omega) \to 0$.

Proof: To prove this proposition, choose $\epsilon > 0$ and let $N_\epsilon = \{v \in \mathbf{S}^{n-1} : \sigma(v) < \epsilon\}$, $U_\epsilon = \{v \in \mathbf{S}^{n-1} : \sigma(v) > 2\epsilon\}$. Then there exists $\delta > 0$ depending only on ϵ such that $\Gamma \cap B_p(\delta) = \{x \in \Gamma : dist_g(x,p) < \delta\}$ is contained in $N_\epsilon(\delta) = \{x = \exp_p(rv) : 0 \leq r < \delta,\ v \in N_\epsilon\}$. It then follows for $x \in U_\epsilon$ that there exists positive constants C_1, C_2 independent of ϵ and δ with

$$C_1\, r\, \sigma(v) \leq dist_g(\exp_p(rv), N_\epsilon(\delta)) \leq C_2\, r\, \sigma(v)$$

and

$$C_1\, \rho(\exp_p(rv)) \leq dist_g(\exp_p(rv), N_\epsilon(\delta)) \leq C_2\, \rho(\exp_p(rv)).$$

Thus, it follows that $r\sigma(v) \approx \rho(\exp_p(rv)$ as $r \to 0$, and therefore that

$$r^{\beta_0}\, u(r,\omega) \approx \sigma(\omega)^{-\beta_0}$$

for $\omega \in \Omega$. Passing through the limits, we see that

$$\overline{v}(\omega) \approx \underline{v}(\omega) \approx \sigma(\omega)^{-\beta_0}$$

which complete the proof of the proposition. $\square$

Proof of Theorem A: By definition of $f \approx g$ and $\overline{v}, \underline{v}$, there exists a small positive constant c such that

$$c\overline{v}(\omega) \leq \underline{v}(\omega) \leq \overline{v}(\omega) \quad \text{for all} \quad \omega \in \mathbf{S}^{n-1} \backslash L_p\Gamma.$$

We also have $c\overline{v}$ to be lower solution to $\Delta_\omega v = v^q + \beta_0 d_0 v$, so the method of upper and lower solutions guarantees the existence of a positive solution to $\Delta_g v = v^q + \beta_0 d_0 v$ with a strong singularity at $L_p\Gamma$ that satisfies $c\overline{v} \leq v \leq \underline{v}$, and the existence of a positive solution to $\Delta_\omega v = v^q + \beta_0 d_0 v$ with a strong singularity at $L_p\Gamma$ with $v \geq \overline{v}$. The uniqueness of solutions with a strong singularity (Lemma 5) then shows that $\overline{v} \equiv \underline{v}$, and thus the limit

$$v(\omega) = \lim_{r\to 0} \, r^{\beta_0} \, u(r,\omega)$$

exists and is the unique positive solution of $\Delta_\omega v = v^q + \beta_0 d_0 v$ in $\mathbf{S}^{n-1}\backslash L_p\Gamma$ with a strong singularity at $L_p\Gamma$. The localization of Aviles' nonexistence result then shows that the local dimension of $L_p\Gamma$ must be greater than $d_0 - 1$, and thus the dimension of the tangent cone must be greater than d_0. $\square$

Let us take a moment to discuss the existence of a solution to (†) with a strong singularity. This is discussed in detail in [Fn2], and the outline for the procedure is contained in [Fn1]. For existence, we demand that the cone of tangent vectors of Γ at p is equal to the tangent cone of Γ at p for all p. Further, we demand that the tangent cones are compatible for two points in the same stratification, and lastly we demand that we can replace a portion of Γ near p with the tangent cone of Γ at p. A modification of the methods in [Fn1] allows us to construct a solution to $\Delta_g u = u^q + Su$ that is singular at the high dimensional portion of $T_p\Gamma$ (for strata in $T_p\Gamma$ with dimension greater than d_0). To conclude that $u(x) \to +\infty$ as $x \to p$, we further require that a high-dimensional piece of $T_p\Gamma$ is cone-like, that is of the form

$$\{(x_1, \cdots, x_m, 0, \cdots, 0) : x_1^2 + \cdots + x_k^2 = x_m^2, \ x_m \geq 0\} \subset \mathbf{R}^n$$

where $k < m \leq n$, with $m > d_0$. This last condition is necessary so we may apply the *continuation argument* developed in [Fn1].

6. Proof of Main Result

The result of the previous section shows that the dimension of every proper tangent cone must be greater than $d - 0$ for the existence of a positive solution to (†) with a strong singularity at Γ. We would like to claim that if the maximal solution u to (†) does not have a strong singularity then the conformal metric $u^{q-1}\, g$ is not complete. But, this is difficult to prove. Instead, we will show that if the dimension of $T_p\Gamma$ is less than d_0, then the metric $\hat{g}$ is not complete. The critical case where the dimension of a proper tangent cone is equal to d_0 requires further study; our proof breaks down exactly in this case.

Theorem B: *Suppose Γ has a proper tangent cone at p with the dimension of this tangent cone less than d_0, and let u be the maximal positive solution to (†). Then the conformal metric $\hat{g} = u^{q-1}\, g$ is not a complete metric on $M\backslash\Gamma$.*

Proof: It suffices to show when $d_0 > 0$ that there exists an $\epsilon > 0$ such that every positive solution to (†) satisfies

$$r^{\beta_0}\, u(r,\omega) = \mathcal{O}(r^{\epsilon}) \quad \text{as} \quad r \to 0 \tag{6.1}$$

for all $\omega \in \mathbf{S}^{n-1}\backslash L_p\Gamma$. Theorem A shows that

$$r^{\beta_0} u(r,\omega) \to 0 \quad \text{as} \quad r \to 0, \tag{6.2}$$

since $\dim(T_p\Gamma) < d_0$. To improve (6.2) to (6.1), we first show that there exists a positive function f such that

$$r^{\beta_0}\, u(r,\omega) \approx f(r) \quad \text{as} \quad r \to 0, \tag{6.3}$$

with f satisfying $f(r) \to 0$ as $r \to 0$, and

$$\lim_{r\to 0} \frac{r f'(r)}{f(r)} \quad \text{and} \quad \lim_{r\to 0} \frac{r^2 f''(r)}{f(r)} \quad \text{both exist.}$$

As a result of these conditions, we find that

$$\overline{v}(\omega) = \limsup_{r\to 0} \frac{r^{\beta_0}\, u(r,\omega)}{f(r)} \tag{6.4}$$

is a positive function on $\mathbf{S}^{n-1}\backslash L_p\Gamma$. A simple modification of the arguments in the proof of Propositions 1 in Section 5 shows that $\overline{v} \in C^{0,\alpha}(\mathbf{S}^{n-1}\backslash L_p\Gamma)$.

We prove the existence of such a function f as follows. First choose $\omega_0 \in \mathbf{S}^{n-1}\backslash L_p\Gamma$, and define

$$f_0(t) = \sup_{r\in[0,t]} r^{\beta_0}\, u(r,\omega_0).$$

Notice, we have defined f_0 so that (6.3) is satisfied when $\omega = \omega_0$. The Harnack inequality, Lemma 3 in Section 4, then guarantees that (6.3) is satisfies for all $\omega in \mathbf{S}^{n-1}\backslash L_p\Gamma$. Notice we have defined f_0 so that it is Lipschitz continuous, and hence is differentiable almost everywhere. Choose a C^1 function f_1 close to f_0 in the sup-norm. By computing the derivative of f_0 when possible, we find that either

$$f_0'(r) = \partial_r(r^{\beta_0}\, u(r,\omega_0))$$

or $f_0'(r) = 0$, so the derivative estimates in Lemma 4 imply that

$$0 \leq \frac{r\, f_0'(r)}{f_0(r)} < C$$

when $f_0'(r)$ is defined. Therefore, we can arrange that

$$0 < \frac{r\, f_1'(r)}{f_1(r)} < C. \tag{6.5}$$

by choosing f_1 to be strictly increasing. We may change the C^1 approximation to be either concave up or concave down without altering the truth of (6.3) or (6.5). Thus, we can choose f_1 such that

$$\lim_{r\to 0} \frac{r\, f_1'(r)}{f_1(r)} \quad \text{exists.} \tag{6.6}$$

Further, we may choose f_1 such that $rf_1'(r)/f_1(r)$ is either monotonically increasing or monotonically decreasing without altering the truth of (6.3), (6.5), or (6.6). Thus, by standard results (cf. [R]), we find that $rf_1'(r)/f_1(r)$ is differentiable almost everywhere. Repeating the arguments in proceeding from f_0 to f_1, we may thus choose a C^2 function f satisfying the (6.3), (6.5), and (6.6) with

$$\left|\frac{r^2 f''(r)}{f(r)}\right| < C. \tag{6.7}$$

We may further choose $f''(r)$ to be monotone without altering the truth of (6.3), (6.5), (6.6) or (6.7) to ensure that

$$\lim_{r \to 0} \frac{r^2 f''(r)}{f(r)} \quad \text{exists.} \tag{6.8}$$

Therefore, we have the existence of the desired function f.

We now consider two cases in proving that (6.2) may be improved to (6.1):

$$\text{(i)} \ \lim_{r\to 0} \frac{r f'(r)}{f(r)} = \epsilon > 0 \quad \text{and} \quad \text{(ii)} \ \lim_{r\to 0} \frac{r f'(r)}{f(r)} = 0.$$

In the first case, a comparison theorem implies that $f(r) \le C\,r^{\epsilon}$, and thus $u(r,\omega) \le Cr^{-\beta_0+\epsilon}$ which implies that $u^{q-1} \le r^{-2+\epsilon'}$ and hence noncompleteness of the metric $u^{q-1}\,g$. In the second case, we find that

$$\lim_{r\to 0} \frac{r^2 f''(r)}{f(r)} = 0,$$

since if we assume

$$\lim_{r\to 0} \frac{r^2 f''(r)}{f(r)} = \epsilon \neq 0 \quad \text{and} \quad \lim_{r\to 0} \frac{r f'(r)}{f(r)} = 0$$

then

$$\lim_{r\to 0} r \frac{d}{dr} \left(\frac{r f'(r)}{f(r)} \right) = \epsilon$$

which implies

$$\left| \frac{r f'(r)}{f(r)} \right| \approx -\log(r).$$

Thus $\frac{rf'(r)}{f(r)}$ tends to infinity as $r \to 0$ contradicting $\frac{rf'(r)}{f(r)} \to 0$. Define the function $v(r,\omega)$ by setting

$$u(r,\omega) = r^{-\beta_0}\, f(r)\, v(r,\omega)$$

Computing $\Delta_g u$ using (5.1), we find that

$$\begin{aligned} \Delta_g u = &-\beta_0 d_0\, r^{-\beta_0-2}\, f\, v + (n-1-2\beta_0) r^{-\beta_0-1}\, f'\, v \\ &+ (n-1-2\beta_0)\, r^{-\beta_0-1}\, f\, \partial_r v + 2r^{-\beta_0}\, f'\, \partial_r v + r^{-\beta_0}\, f''\, v \\ &+ r^{-\beta_0}\, f\, \partial_r^2 v + r^{-\beta_0-2}\, f\, \Delta_\omega v + \mathcal{L}(r^{-\beta_0}\, f\, v). \end{aligned}$$

We may thus write $\Delta_g u = u^q + Su$ in terms of the functions f and v as

$$
\begin{aligned}
&-\beta_0 d_0\, v + (n-1-2\beta_0)\frac{rf'}{f}\, v + (n-1-2\beta_0)\, r\,\partial_r v + 2\frac{r^2 f'}{f}\partial_r v \\
&+\frac{r^2 f''}{f}\, v + r^2\partial_r^2 v + \Delta_\omega v = f^{q-1}\, v^q + \mathcal{O}(r),
\end{aligned}
$$

since the derivative estimates and the form of the perturbation operator $\mathcal{L}$ imply

$$
\frac{r^{\beta_0 q}}{f}\,\mathcal{L}(r^{-\beta_0}\, f\, v) = \mathcal{O}(r).
$$

Using the change of variables $r = e^{-t}$ and applying the one-dimensional version of the asymptotic maximum principle along with estimates on f', f'', and v then allows us to conclude that $\overline{v}(\omega)$ weakly satisfies $\Delta_\omega v \geq \beta_0 d_0\, v$ on $\mathbf{S}^{n-1}\backslash L_p\Gamma$. (Repeat the arguments in the proof of Proposition 2 in Section 5, and consult [FM] for the details of the change of variables.)

To conclude $f(r) \leq C\, r^\epsilon$, we compare $\overline{v}$ to a solution of the linear equation $\Delta_\omega v = \beta_0 d_0 v$ on $\mathbf{S}^{n-1}\backslash L_p\Gamma$ that has a nonremovable singularity at $L_p\Gamma$. We prove the existence of such a solution by using the Poisson transform developed in [Fn1]. Let $\{S_\alpha\}$ be a stratification of $L_p\Gamma$ (exists since $T_p\Gamma$ is a proper tangent cone), and let $\varphi_\alpha \in C(S_\alpha)\cap L^1(S_\alpha)$ be positive on S_α. (We use the induced metric on S_α to define a measure μ_α on S_α.) We now construct a solution on $S^{n-1}\backslash L_p\Gamma$ (or a measure on $L_p\Gamma$ - these are equivalent cf. [Mz]) by using the linearity of the equation to add solutions

$$
v_\alpha(x) = \int\limits_{\overline{S_\alpha}} \mathcal{G}(x,y)\,\varphi_\alpha(y)\, d\mu_\alpha(y) \tag{6.9}
$$

to $\Delta_\omega v = \beta_0 d_0\, v$ on $\mathbf{S}^{n-1}\backslash S_\alpha$ where $\mathcal{G}(x,y)$ is the Green's function for $-\Delta_\omega + \beta_0 d_0$. (The positivity of φ_α ensures that v_α is positive.) A simple estimation of the integral in (6.9) (cf. [Fn1]) shows that as $x \to \mathrm{int}(S_\alpha)$ that

$$
\sigma(x)^{n-k-3}\, v_\alpha(x) \geq \epsilon \quad \text{if} \quad k < n-3
$$

or

$$
\big(1 + \big|\log(\sigma(x)\big|\big)^{-1}\, v_\alpha(x) \geq \epsilon \quad \text{if} \quad k = n-3
$$

where k is the dimension of S_α. Notice that the assumption $\dim(T_p\Gamma) = \dim(L_p\Gamma)+1 < d_0$ implies $k < n-3$.

We now claim there exists a positive solution v of $(*)$ such that $\overline{v} \geq v$ on a path tending towards $L_p\Gamma$. By the linearity of the equation, we can clearly choose a positive solution v of $(*)$ such that for a given point $w_0 \in \mathbf{S}^{n-1}\backslash L_p\Gamma$ we have $\overline{v}(\omega_0) > v(\omega_0)$. The difference function $w = \overline{v} - v$ then satisfies $w(\omega_0) > 0$ and in the set $\Omega = \{\omega \in \mathbf{S}^{n-1}\backslash L_p\Gamma : w > 0\}$ we have $\Delta_\omega w \geq 0$, so the maximum principle then implies that $\sup_{\partial\Omega} w > 0$. We can therefore conclude that there exists a path on $\mathbf{S}^{n-1}\backslash L_p\Gamma$ tending to $L_p\Gamma$ with $\overline{v} \geq v$, and we may also assume that $\overline{v} \geq \delta\,\sigma^{-(n-k-3)}$ for some positive constant δ where $k = \dim(L_p\Gamma)$. Hence, $u(r,\omega) \geq \delta\, r^{-\beta_0}\, f(r)\, \sigma^{-(n-k-3)}$ close to p. Choosing σ proportional to r, and using the fact that $\rho \approx \sigma\, r$ (see the proof of Proposition 3 in Section 5), we find that

$$u \geq C\, r^{-\beta_0-(n-k-3)}\, f(r)$$

on a path in $M\backslash\Gamma$ approaching p. But, from the upper bound in Lemma 1, we know that $u \leq C\rho^{-\beta_0}$ on this path which implies that $f(r)$ must satisfy

$$f(r)\, r^{-\beta_0-(n-k-3)} \leq C\, r^{-2\beta_0},$$

or

$$f(r) \leq C\, r^{-\beta_0+n-k-3} \leq C\, r^{d_0-k-1}.$$

The assumption that $\dim(T_p\Gamma) < d_0$ then implies that $k < d_0 - 1$ and thus $f(r) \leq C\, r^\epsilon$. From which, a simple calculation shows $\hat{g} = u^{q-1}\, g$ is not complete. $\square$

Remark: The proof breaks down exactly when $k = d_0 - 1$. In this case, we can conclude only that $f(r) \leq C$. A possibly cause why our proof breaks down in this case is that we do not strongly use the "negative case" in this last part of the proof; in the positive case (cf. [Pa]) when $q = \frac{n+2}{n-2}$ there exist solutions when $\dim(\Gamma) = d_0$ where the asymptotic growth is on the order of

$$u \approx r^{-(n-2)/2}\, \log(1/r)^{-(n-2)/4}.$$

And moreover, this type of growth implies the conformal metric $\hat{g} = u^{4/(n-2)}\, g$ is a complete metric.

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